Auxiliar 1

Representaciones del grupo de Lorentz

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P1.-

a) Considerando una transformación infinitesimal $\Lambda = 1 + \delta\omega$ y la transformación de un campo escalar

$$U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x),$$

muestre que

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x),$$

donde

$$\mathcal{L}^{\mu\nu} \equiv \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

y $M^{\mu\nu}$ son los generadores del grupo de Lorentz.

b) Muestre que

$$[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x).$$

c) Use el resultado anterior y la identidad de Jacobi

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

para demostrar que

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x).$$

P2.-

a) Nuevamente consideremos $\Lambda = 1 + \delta\omega$, pero para la transformación de un *campo vectorial*

$$U^{-1}(\Lambda)\partial^\mu\varphi(x)U(\Lambda) = \Lambda^\mu{}_\rho\bar{\partial}^\rho\varphi(\Lambda^{-1}x).$$

Demuestre que

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_{\mathrm{V}}^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x),$$

donde

$$(S_{\mathrm{V}}^{\mu\nu})^\rho{}_\tau \equiv \frac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)$$

b) Muestre que las matrices $S_{\mathrm{V}}^{\mu\nu}$ deben seguir las mismas relaciones de conmutación que los operadores $M^{\mu\nu}$,

$$\Big[M^{\mu\nu}, M^{\alpha\beta}\Big] = i\hbar\Big(g^{\mu\alpha}M^{\nu\beta} - (\mu \leftrightarrow \nu)\Big) - (\alpha \leftrightarrow \beta).$$

c) Para una rotación con un ángulo θ alrededor del eje z, tenemos la transformación de Lorentz

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Muestre que

$$\Lambda = \exp\Big(-i\theta S_{\mathrm{V}}^{12}/\hbar\Big).$$

d) Mientras que para un *boost* con *rapidity* η en la dirección de z, tenemos

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}.$$

Muestre que

$$\Lambda = \exp\Big(+i\eta S_{\mathrm{V}}^{30}/\hbar\Big).$$

P3.-

Inspirado en las transformaciones de campos escalares y vectoriales, escribamos la transformación de un campo con un índice de Lorentz arbitrario $\psi_A(x)$, dada por

$$U^{-1}(\Lambda)\psi_A(x)U(\Lambda) = L_A{}^B(\Lambda)\psi_B(x),$$

donde $L_A{}^B(\Lambda)$ es una función de Λ, por lo que considerando $\Lambda = 1 + \delta\omega$ podemos expandir a primer orden en $\delta\omega$ como

$$L_A{}^B(\Lambda) = \delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}(S_V^{\mu\nu})_A{}^B$$

al igual que con $U(\Lambda)$

$$U(\Lambda) = I + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}.$$

Muestre que

$$[\psi_A(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_A(x) + (S_V^{\mu\nu})_A{}^B\psi_B(x),$$

donde, hasta este punto, $S_V^{\mu\nu}$ no tiene ninguna expresión en particular. En un futuro, estos operadores $S_V^{\mu\nu}$ nos permitirán definir las teorías de partículas con spin mayor a 0, por ejemplo fermiones en *Quantum Electrodynamics*, QED.

Auxiliar 1

P1

a) Nos piden considerar una pequeña transformación de Lorentz $\Lambda = 1 + \delta\omega$, que escrito como elementos de una matriz de 4×4 (para un espacio-tiempo $3+1$-dimensional)

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}, \qquad \mu,\nu \in \{0,1,2,3\}.$$

Para actuar sobre un espacio de Hilbert donde actúen nuestros campos cuánticos, debemos hacer un mapeo de los elementos $\Lambda \in SO^{+}(3,1)$ al espacio de Hilbert a través de un operador lineal unitario $U(\Lambda)$.

Sabemos que $U(\Lambda)$ actua sobre $\varphi(x)$ como

$$\varphi(x) \to U^{-1}(\Lambda)\,\varphi(x)\,U(\Lambda) = \varphi(\Lambda^{-1}x) \quad (1)$$

donde podemos considerar $\Lambda = 1 + \delta\omega$ y expandir en serie de Taylor ambos lados de (1) hasta orden lineal en $\delta\omega$. Expandiendo $U(\Lambda)$

$$U(\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^{n} U(\Lambda)}{\partial(\Lambda_{\mu\nu})^{n}}\bigg|_{\delta\omega=0} \delta\omega_{\mu\nu}$$

$$= 1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu} M^{\mu\nu} + \cancel{\mathcal{O}(\delta\omega^{2})}^{\sim 0}, \quad \text{donde} \quad M^{\mu\nu} \equiv \frac{2\hbar}{i}\frac{\partial U(\Lambda)}{\partial \Lambda_{\mu\nu}}\bigg|_{\delta\omega=0}$$

Además, sabemos que $\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu} \in \mathbb{R}$ y como U es unitario, $U^{-1} = U^{\dagger}$, entonces $M^{\mu\nu} = -M^{\nu\mu}$ y hermítico $M^{\dagger} = M$. Por lo tanto,

$$U^{-1}(\Lambda) = 1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu} M^{\mu\nu} + \cancel{\mathcal{O}(\delta\omega^{2})}^{\sim 0}$$

Reemplazando en LHS de (1)

$$\Rightarrow U^{-1}(\Lambda)\varphi(x)U(\Lambda) = \left[1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right]\varphi(x)\left[1 + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right]$$

$$= 1\,\varphi(x) - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\varphi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}\varphi(x)M^{\mu\nu} + \cancel{\mathcal{O}(\delta\omega^{2})}^{\sim 0}$$

$$= \varphi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}[\varphi(x), M^{\mu\nu}] \quad (2)$$

donde solo los generadores $M^{\mu\nu}$ actúan sobre $\varphi(x)$, $\delta\omega$ son solo c-numbers que podemos mover de un lado a otro c/r a los operadores (al ser lineales)

Expandamos el RHS de (1) hasta orden $\delta\omega$,

$$\varphi(\Lambda^{-1}x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \varphi(\Lambda^{-1}x)}{\partial((\Lambda^{-1}x)^\mu)^n}\Bigg|_{\delta x=0} (\delta x^\mu)^n$$

$$= \varphi(x) + \delta x^\mu \frac{\partial \varphi(x)}{\partial x^\mu} + \cancel{\mathcal{O}(\delta x^2)}^{\sim 0} \quad (3)$$

donde $(\Lambda^{-1}x)^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu = \Lambda_\nu{}^\mu x^\nu = (\delta_\nu{}^\mu + \delta\omega_\nu{}^\mu) x^\nu$

$$= x^\mu + \delta\omega_\nu{}^\mu x^\nu \equiv x^\mu + \delta x^\mu$$

así que reemplazando en (3)

$$\varphi(\Lambda^{-1}x) = \varphi(x) + \delta\omega_\nu{}^\mu x^\nu \partial_\mu \varphi(x)$$

$$= \varphi(x) + g^{\mu\alpha} \delta\omega_{\nu\alpha} x^\nu \partial_\mu \varphi(x)$$

$$= \varphi(x) + \delta\omega_{\nu\alpha} x^\nu \partial^\alpha \varphi(x)$$

$$= \varphi(x) + \delta\omega_{\mu\nu} x^\mu \partial^\nu \varphi(x) , \quad \delta\omega_{\mu\nu} x^\mu \partial^\nu \varphi = \delta\omega_{\nu\mu} x^\nu \partial^\mu \varphi = -\delta\omega_{\mu\nu} x^\nu \partial^\mu \varphi$$

$$= \varphi(x) + \frac{\delta\omega_{\mu\nu}}{2} (x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi(x) \quad (4)$$

Juntando (2) y (4)

$$\cancel{\varphi(x)} + \frac{i}{2\hbar} \delta\omega_{\mu\nu} [\varphi(x), M^{\mu\nu}] = \cancel{\varphi(x)} + \frac{\delta\omega_{\mu\nu}}{2} (x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi(x)$$

$$\therefore [\varphi(x), M^{\mu\nu}] = \frac{\hbar}{i} (x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi(x) \equiv \mathcal{L}^{\mu\nu} \varphi(x) \quad \square$$

b) Debido a que $M^{\mu\nu}$ depende de la transf. de Lorentz Λ, y esta es ind. de las coordenadas x^μ, tenemos que

$$[\mathcal{L}^{\mu\nu}, M^{\mu\nu}] = 0$$

al ser $\mathcal{L}^{\mu\nu}$ un operador en función de las coord.. Entonces

$$\mathcal{L}^{\rho\sigma} \varphi(x) = [\varphi(x), M^{\rho\sigma}] \quad / \mathcal{L}^{\mu\nu}.$$

$$\Rightarrow \mathcal{L}^{\mu\nu} \mathcal{L}^{\rho\sigma} \varphi(x) = \mathcal{L}^{\mu\nu} [\varphi(x), M^{\rho\sigma}]$$

$$= [\mathcal{L}^{\mu\nu} \varphi(x), M^{\rho\sigma}]$$

$$= [[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] \quad \square$$

c) Usaremos que $[C,[A,B]] = [[B,C],A] + [[C,A],B]$ con

$$C = \varphi(x), \quad A = M^{\mu\nu}, \quad B = M^{\rho\sigma}$$

$$\Rightarrow [\varphi(x),[M^{\mu\nu},M^{\rho\sigma}]] = [[M^{\rho\sigma},\varphi(x)],M^{\mu\nu}] + [[\varphi(x),M^{\mu\nu}],M^{\rho\sigma}]$$

$$= -\mathcal{L}^{\rho\sigma}[\varphi(x),M^{\mu\nu}] + \mathcal{L}^{\mu\nu}[\varphi(x),M^{\rho\sigma}]$$

$$= -\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\varphi(x) + \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\varphi(x)$$

$$= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\varphi(x) \quad \square$$

P2

a) Queremos encontrar la expresión de $[\partial^\rho \varphi(x), M^{\mu\nu}]$, así que usaremos que sabemos

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x)$$

donde podemos actuar por la izq. con ∂^ρ notando que $[\partial^\rho, M^{\mu\nu}]=0$ (mismo argumento de antes)

$$\Rightarrow [\partial^\rho\varphi(x), M^{\mu\nu}] = \partial^\rho \mathcal{L}^{\mu\nu}\varphi(x)$$

$$= \frac{\hbar}{i}[\partial^\rho(x^\mu\partial^\nu) - \partial^\rho(x^\nu\partial^\mu)]\varphi(x)$$

$$= \frac{\hbar}{i}[(\partial^\rho x^\mu)\partial^\nu + x^\mu\partial^\rho\partial^\nu - (\partial^\rho x^\nu)\partial^\mu - x^\nu\partial^\rho\partial^\mu]\varphi(x)$$

donde $\partial^\alpha x^\beta = g^{\alpha\gamma}\partial_\gamma x^\beta = g^{\alpha\gamma}\delta_\gamma{}^\beta$ y las derivadas parciales conmutan $\partial^\alpha\partial^\beta = \partial^\beta\partial^\alpha$

$$\Rightarrow [\partial^\rho\varphi, M^{\mu\nu}] = \frac{\hbar}{i}[x^\mu\partial^\nu - x^\nu\partial^\mu]\partial^\rho\varphi(x) + \frac{\hbar}{i}[g^{\rho\gamma}\delta_\gamma{}^\mu\partial^\nu - g^{\rho\gamma}\delta_\gamma{}^\nu\partial^\mu]\varphi(x)$$

$$= \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + \frac{\hbar}{i}[g^{\rho\mu}\partial^\nu - g^{\rho\nu}\partial^\mu]\varphi(x)$$

$$= \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + \frac{\hbar}{i}[g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau]\partial^\tau\varphi(x)$$

$$= \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_V^{\mu\nu})^\rho{}_\tau\,\partial^\tau\varphi(x) \quad \square$$

b) Calculemos el conmutador $[S_V^{\mu\nu}, S_V^{\alpha\beta}]$

$$[S_V^{\mu\nu}, S_V^{\alpha\beta}]^\rho{}_\tau = (S^{\mu\nu}\cdot S^{\alpha\beta})^\rho{}_\tau - (S^{\alpha\beta}\cdot S^{\mu\nu})^\rho{}_\tau$$

$$= (S^{\mu\nu})^\rho{}_\gamma (S^{\alpha\beta})^\gamma{}_\tau - (S^{\alpha\beta})^\rho{}_\gamma(S^{\mu\nu})^\gamma{}_\tau$$

$$= \left(\frac{\hbar}{i}\right)^2 (g^{\mu\rho}\delta^\nu{}_\gamma - g^{\nu\rho}\delta^\mu{}_\gamma)(g^{\alpha\gamma}\delta^\beta{}_\tau - g^{\beta\gamma}\delta^\alpha{}_\tau) - (\alpha\leftrightarrow\mu,\ \beta\leftrightarrow\nu)$$

$$= \left(\frac{\hbar}{i}\right)^2 (g^{\mu\rho}g^{\alpha\gamma}\delta^\nu{}_\gamma\delta^\beta{}_\tau - g^{\mu\rho}g^{\beta\gamma}\delta^\nu{}_\gamma\delta^\alpha{}_\tau - g^{\nu\rho}g^{\alpha\gamma}\delta^\mu{}_\gamma\delta^\beta{}_\tau + g^{\nu\rho}g^{\beta\gamma}\delta^\mu{}_\gamma\delta^\alpha{}_\tau) - (\%)$$

$$= \left(\frac{\hbar}{i}\right)^2 \left[(g^{\mu\rho}g^{\alpha\nu}\delta^\beta{}_\tau - g^{\mu\rho}g^{\beta\nu}\delta^\alpha{}_\tau) - (g^{\nu\rho}g^{\alpha\mu}\delta^\beta{}_\tau - g^{\nu\rho}g^{\beta\mu}\delta^\alpha{}_\tau)\right] - (\%)$$

$$= \frac{\hbar}{i}\left(g^{\mu\rho}(S_V^{\alpha\beta})^\nu{}_\tau - g^{\nu\rho}(S_V^{\alpha\beta})^\mu{}_\tau\right) - \frac{\hbar}{i}\left(g^{\alpha\rho}(S^{\mu\nu})^\beta{}_\tau - g^{\beta\rho}(S_V^{\mu\nu})^\alpha{}_\tau\right)$$

$$= \frac{\hbar}{i}\left(g^{\mu\rho}(S_V^{\alpha\beta})^\nu{}_\tau - (\mu\leftrightarrow\nu)\right) - (\alpha\leftrightarrow\beta)$$

que es el mismo de M,

$$[M^{\mu\nu}, M^{\alpha\beta}] = i\hbar\left(g^{\mu\alpha}M^{\nu\beta} - (\mu \leftrightarrow \nu)\right) - (\alpha \leftrightarrow \beta)$$

c) Sabemos que $(S_V^{12})^\rho{}_\tau = \frac{\hbar}{i}\left(g^{1\rho}\delta^2{}_\tau - g^{2\rho}\delta^1{}_\tau\right)$

$$= \frac{\hbar}{i} g^{1\rho}\delta^2{}_\tau - \frac{\hbar}{i} g^{2\rho}\delta^1{}_\tau$$

donde el primer término es distinto de 0 para $\rho = 1 \wedge \tau = 2$ y el segundo término en $\rho = 2 \wedge \tau = 1$. Matricialmente, esto es

$$(S_V^{12})^\rho{}_\tau = \frac{\hbar}{i} \begin{array}{cc} \begin{array}{cccc} \tau=0 & \tau=1 & \tau=2 & \tau=3 \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \rho=0 \\ \rho=1 \\ \rho=2 \\ \rho=3 \end{array} \end{array} \equiv \frac{\hbar}{i} A^\rho{}_\tau$$

que si reemplazamos en Λ

$$\Lambda = \exp(-i\theta S_V^{12}/\hbar)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\theta S_V^{12}}{\hbar}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(-\theta A)^n = 1 - \theta A + \frac{1}{2!}\theta^2 A^2 - \frac{1}{3!}\theta^3 A^3 + \ldots$$

donde $A \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv -B$ y $A \cdot A \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -A$

Entonces, $A^{2n} = (-1)^n B$ y $A^{2n+1} = (-1)^n A$

$$\Rightarrow \Lambda = \sum_{n=0}^{\infty} \frac{1}{n!}(-\theta A)^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-\theta A)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-\theta A)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-\theta)^{2n}(-1)^n B + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-\theta)^{2n+1}(-1)^n A$$

$$= B \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} - A \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$$

$$= B\cos\theta - A\sin\theta$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \Lambda^\mu{}_\nu$$

d) Ahora analicemos S^{30}_{ν}

$$(S^{30}_{\nu})^{\rho}{}_{\tau} = \frac{\hbar}{i}\left(g^{3\rho}\delta^{0}{}_{\tau} - g^{0\rho}\delta^{3}{}_{\tau}\right) = \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \equiv \frac{\hbar}{i}\tilde{A}$$

$* g^{00} = -1 \,,\; g^{ii} = 1$

donde $\tilde{A}^{2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \tilde{B}$ y $\tilde{A}^{3} = \tilde{A}$ $\therefore \tilde{A}^{2n} = \tilde{B} \wedge \tilde{A}^{2n+1} = \tilde{A}$

entonces $\Lambda = \exp\left(\frac{i\eta S^{30}_{\nu}}{\hbar}\right) = \sum_{n=0}^{\infty} \frac{\eta^{2n}}{(2n)!}\tilde{A}^{2n} + \sum_{n=0}^{\infty} \frac{\eta^{2n+1}}{(2n+1)!}\tilde{A}^{2n+1} = \tilde{B}\cosh(\eta) + \tilde{A}\sinh(\eta) = \Lambda^{\mu}{}_{\nu}$

P3

Consideremos la transformación

$$U^{-1}(\Lambda)\psi_A(x)U(\Lambda) = L_A{}^B(\Lambda)\psi_B(\Lambda^{-1}x)$$

tomando $\Lambda = 1 + \delta\omega$. El LHS sería

$$U^{-1}(\Lambda)\psi_A(x)U(\Lambda) = \psi_A(x) + \frac{i}{2}\delta\omega_{\mu\nu}[\psi_A(x), M^{\mu\nu}]$$

mientras que el RHS

$$L_A{}^B(\Lambda)\psi_B(\Lambda^{-1}x) = \left[\delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{v})_A{}^B\right]\left[\psi_B(x) + \delta\omega_{\mu\nu}x^\mu\partial^\nu\psi_B(x)\right]$$

$$= \psi_A(x) + \delta\omega_{\mu\nu}x^\mu\partial^\nu\psi_A(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{v})_A{}^B\psi_B(x)$$

$$= \psi_A(x) + \frac{\delta\omega_{\mu\nu}}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi_A(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{v})_A{}^B\psi_B(x)$$

que igualando

$$\Rightarrow \psi_A(x) + \frac{i}{2}\delta\omega_{\mu\nu}[\psi_A(x), M^{\mu\nu}] = \psi_A(x) + \frac{\delta\omega_{\mu\nu}}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi_A(x) + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{v})_A{}^B\psi_B(x)$$

$$\therefore\ [\psi_A(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_A(x) + (S^{\mu\nu}_{v})_A{}^B\psi_B(x)$$