

# Regularizing infrared divergences in de Sitter spacetime

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Correlation functions of light scalars in de Sitter space, computed in standard perturbation theory, are hindered by time-dependent infrared divergences in the form of powers of  $\ln a(t)$ , where  $a(t)$  is the scale factor describing the expansion of space. It has often been pointed out that loop corrections to these correlation functions make their divergence even stronger. In this note, we argue that this is not the case: Loop corrections can be treated systematically with standard perturbative techniques (such as dimensional regularization) without necessarily introducing new  $\ln a(t)$  dependencies. To be concrete, we focus on correlation functions represented by diagrams with a single vertex and an arbitrary number of loops. In this case, divergences from loops can be removed systematically with counterterms order by order, and one finds that observable loop-corrected correlation functions are indistinguishable from their tree-level form. By adopting a Wilsonian perspective, we further point out that our results favor the use of physical cutoffs (as opposed to comoving cutoffs) to regularize infrared divergences in general diagrams with an arbitrary number of loops and vertices.

## I. INTRODUCTION

The aim of this note is to comment on the status of loop corrections to  $n$ -point correlation functions of light scalar fields in de Sitter space [1–38]. Non-derivative interactions—those descending from scalar field potentials—lead to the appearance of infrared (IR) asymptotic divergences in light-field correlation functions [1–4], in the form of powers of  $\ln a(t)$ , where  $a(t)$  is the scale factor parametrizing the expansion of flat spatial slices. Already at tree level,  $n$ -point correlation functions in momentum space turn out to be proportional to  $[\ln a(t)]^V$ , with  $V$  the number of vertices participating in a contributing diagram [20, 29]. Do loop corrections alter this behavior? The answer to this question is often found to be positive [1–26], resulting from the way in which loop integrals are dealt with. We wish to revisit this argument.

It is well known that massless scalar fields do not allow the construction of normalizable de Sitter-invariant vacuum states [39, 40]. This issue arises from the lack of a gap between the vacuum state and zero-momentum multi-particle states, which, in turn, causes divergences in propagators (written in coordinate space) due to unbounded zero-mode contributions. Let us recall that propagators participate in the computation of correlation functions in two distinct ways: as bulk-to-boundary propagators and as bulk-to-bulk propagators. In the former class, infrared divergences are not a concern. This is simply because the range of wavelengths to which we

have observational access is finite, which, in particular, excludes zero modes. In practice, correlation functions participating in the computation of observables must be filtered via window functions restricting the range of external momenta flowing into the contributing diagrams. Depending on the case under consideration, this filtering can be implemented in different ways. A typical choice consists of screening bulk-to-boundary propagators from infinitely long modes with a window function featuring a comoving infrared momentum cutoff.

On the contrary, infrared divergences emerging from bulk-to-bulk propagators are a real concern, as they lead to the appearance of unbounded loop integrals exhibiting both infrared and ultraviolet (UV) divergences. A common regularization method is the introduction of cutoffs. The main purpose of this note is to highlight the nature and implications of the infrared momentum cutoff used to regularize these loop corrections. If this cutoff is set as a constant in comoving coordinates [1–26], like the one mentioned above to restrict the external legs of a diagram,<sup>1</sup> the resulting propagator becomes finite but contributes extra time dependencies to correlators [8], now scaling as  $[\ln a(t)]^{V+L}$ . Here,  $L$  represents the loop number in a diagram with  $V$  vertices. Conversely, if the infrared cutoff is constant in physical coordinates [27–33], the internal propagators remain finite while correlators maintain their tree-level temporal behavior of  $[\ln a(t)]^V$ .

It's worth noting a prevailing rationale favoring the adoption of comoving cutoffs: the notion that inflation must possess an origin [1–4], and, therefore, a comoving boundary signaling the physical size of the expand-

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<sup>1</sup> Recall that physical momentum  $\mathbf{p}$  is related to comoving momentum  $\mathbf{k}$  via  $\mathbf{p} = \mathbf{k}/a(t)$ . Hence, a constant comoving cutoff implies a time-dependent physical cutoff and vice versa.

ing universe at some initial time must be taken into account in the computation of correlators. While this statement might have merit, we observe that, sooner or later, this comoving boundary will reach the infrared physical length scale determining the validity of perturbation theory. As such, the details of how the universe looks on scales much larger than this length scale cannot be relevant for the computation of correlation functions using perturbation theory towards the infrared.

Here, we argue that loops, properly treated, cannot increase the infrared divergence of diagrams. Our argument is simple: infrared divergences arising from loop integrals can be regularized without resorting to time-dependent cutoffs in physical momentum space (in accordance with Refs. [31–33]). We will examine this statement in detail for the case of diagrams with one vertex and an arbitrary number of loops, an example that is particularly simple to solve exactly because these loops carry no external momenta. Even though this case might seem too limited to draw useful general conclusions, we notice that in this example we already obtain results that depart from the standard lore where loops contribute to the enhancement of infrared divergences. Moreover, the key elements allowing us to appreciate that this statement should hold true in more general diagrams are already present in the analysis of single-vertex diagrams.

In a nutshell, we will offer a two-pronged analysis to show that infrared divergences appearing in propagators can be safely treated with standard perturbative techniques. First, we show that infrared divergences due to bulk-to-bulk propagators appearing in loops can be absorbed via counterterms within standard schemes such as dimensional regularization. The resulting correlation functions turn out to be finite and independent of any choice of cutoffs (comoving or physical). In the second approach, we adopt a Wilsonian perspective and define an effective theory with infrared and ultraviolet cutoff scales defined in physical momentum space. By requiring that observable correlation functions must remain independent of these scales, we recover the same conditions encountered in the first analysis based on dimensional regularization. All in all, independent of the approach, the resulting effective theory retains the critical property of time-translation invariance—a manifest property of de Sitter spacetime—already present in the starting bare theory. Indeed, using physical momentum cutoffs, the renormalized Lagrangian remains time-translation invariant. On the contrary, the correlation functions and their dynamics attain a time dependence that reflects the spontaneous breaking of the de Sitter isometries by the Bunch-Davies state [39, 40].

To conclude this preamble, let us emphasize that the statement that loops in de Sitter correlators should not be regulated with constant comoving cutoffs in the IR is not new. (See, for instance, Refs. [31–36] for different reasons why such a cutoff may lead to spurious results.) Furthermore, as has been argued in Ref. [37], the distinction between comoving and physical UV cutoffs signifi-

cantly impacts the computation of observables. In this article, we hope to offer new arguments in favor of working with constant physical IR cutoffs regulating loops or, equivalently, resorting to dimensional regularization, which avoids cutoffs.<sup>2</sup>

## II. CORRELATION FUNCTIONS

We will work in a de Sitter background spacetime described by a line element given by  $ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2$ , where  $a(t) \propto e^{Ht}$  is the scale factor and  $H$  is the Hubble scale setting the de Sitter radius. We will also work with conformal time, hereby defined as  $\tau = -1/Ha(t)$ , allowing one to rewrite the metric as  $ds^2 = a^2(\tau) (-d\tau^2 + d\mathbf{x}^2)$ . We shall consider a standard canonical single field theory with a bare action of the form:

$$S = \int d^3x d\tau a^4 \left[ \frac{(\varphi')^2}{2a^2} - \frac{(\nabla\varphi)^2}{2a^2} - \frac{m^2\varphi^2}{2} - \mathcal{V}_b(\varphi) \right], \quad (1)$$

where  $\mathcal{V}_b(\varphi)$  represents an arbitrary bare potential and a prime ( $'$ ) denotes the derivative with respect to  $\tau$ . To perform perturbation theory, we shall treat the entire potential  $\mathcal{V}_b(\varphi)$  as a small perturbative quantity.<sup>3</sup> In addition to  $\mathcal{V}_b(\varphi)$ , we consider a mass parameter  $m$  that will help us to tackle separately the distinct situations where the field is massive and exactly massless. We take the term  $\frac{1}{2}m^2\varphi^2$  as part of the free theory, whereas the quadratic part of the potential  $\frac{1}{2}\lambda_2^b\varphi^2 \subset \mathcal{V}_b(\varphi)$  [see Eq. (5)] is taken to be a small perturbative parameter leading to the definition of two-legged vertices entering the computation of correlation functions.

The general quantized solution for  $\varphi(\mathbf{x}, \tau)$  may be written as

$$\varphi(\mathbf{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \varphi_{\mathbf{k}}(\tau), \quad (2)$$

where  $\varphi_{\mathbf{k}}(\tau)$  is the field in comoving momentum space. At zeroth order ( $\mathcal{V}_b(\varphi) = 0$ ) this quantity may be written as  $\varphi_{\mathbf{k}}(\tau) = f_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}} + f_{\mathbf{k}}^*(\tau)\hat{a}_{-\mathbf{k}}^\dagger$ , where  $\hat{a}_{\mathbf{k}}^\dagger$  and  $\hat{a}_{\mathbf{k}}$  are creation and annihilation operators satisfying  $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ , and  $f_{\mathbf{k}}(\tau)$  represents a mode function satisfying the free-field equations of motion in comoving

<sup>2</sup> Another way to compute correlators is via the wavefunction of the universe approach [41–44]. The result of Ref. [42] for the 1-loop, 2-point function agrees with the result of Refs. [31, 32], which advocate a physical IR cutoff regulating the loop.

<sup>3</sup> This statement will be made more precise once we define the renormalized potential  $\mathcal{V}_{\text{ren}}(\varphi)$  and require that  $|\mathcal{V}_{\text{ren}}(\varphi)| \ll H^4$  for all relevant values of  $\varphi$ . For instance, as we shall later confirm, an axionic bare potential of the form  $\mathcal{V}_b(\varphi) = -A_b \cos(\varphi/f)$  will lead to a renormalized potential  $\mathcal{V}_{\text{ren}}(\varphi) = -A \cos(\varphi/f)$ . Thus, perturbation theory remains valid as long as  $A \ll H^4$ .

momentum space, with Bunch-Davies initial conditions. In the case of a massless scalar field, it is given by

$$f_k(\tau) = \frac{H}{\sqrt{2k^3}}(1 + ik\tau)e^{-ik\tau}, \quad (3)$$

whereas, in the more general case of a massive field, it is given by

$$f_k(\tau) = \frac{i\sqrt{\pi}H}{2k^{3/2}}(-k\tau)^{3/2}H_\nu^{(1)}(-k\tau), \quad (4)$$

where  $H_\nu^{(1)}(x)$  is a Hankel function of the first kind with  $\nu = \sqrt{9/4 - m^2/H^2}$ .

In order to compute correlation functions perturbatively, we expand the bare potential  $\mathcal{V}_b(\varphi)$  in a Taylor series:

$$\mathcal{V}_b(\varphi) = \sum_{n=0}^{\infty} \frac{\lambda_n^b}{n!} \varphi^n, \quad (5)$$

where  $\lambda_n^b$  denote arbitrary bare coupling constants. We may then resort to one of the many procedures available, such as the in-in Hamiltonian scheme [24, 41, 45], the in-in Schwinger-Keldysh approach [46, 47], or the wavefunction of the universe formalism [41–44]. In all of these approaches, the coupling  $\lambda_n^b$  defines an  $n$ -legged vertex allowing the building of diagrams. More to the point, to order  $\lambda$ , an  $n$ -point correlation function in comoving momentum space, evaluated at a time  $\tau$ , can be expressed as the following sum of diagrams with a single vertex:

$$\langle \varphi_{\mathbf{k}_1} \cdots \varphi_{\mathbf{k}_n} \rangle(\tau) = \begin{array}{c} \begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \square \quad \square \quad \dots \quad \square \quad \square \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \lambda_n^b, \bar{\tau} \end{array} \\ + \begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \square \quad \square \quad \dots \quad \square \quad \square \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \lambda_{n+2}^b, \bar{\tau} \end{array} \\ + \begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \square \quad \square \quad \dots \quad \square \quad \square \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \lambda_{n+4}^b, \bar{\tau} \end{array} \\ + \dots \end{array} \quad (6)$$

Note that in the previous expression, an arbitrary diagram with  $n$  external legs and  $L$  loops must be proportional to  $\lambda_{n+2L}^b$  [27]. Using, for instance, Feynman

rules within the Schwinger-Keldysh formalism (see Appendix A), a single diagram with  $L$  loops is found to have the following form:

$$\begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \square \quad \square \quad \dots \quad \square \quad \square \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \lambda_{n+2L}^b, \bar{\tau} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ L \text{ loops} \end{array} = -(2\pi)^3 \delta^{(3)}(\mathbf{K}) 2 \text{Im} \left\{ \frac{\lambda_{n+2L}^b}{2^L L! H^4} \int_{-\infty}^{\tau} \frac{d\bar{\tau}}{\bar{\tau}^4} G_+(\bar{\tau}, k_1) \cdots G_+(\bar{\tau}, k_n) \left[ \int \frac{d^3 k}{(2\pi)^3} G_{++}(\bar{\tau}, \bar{\tau}, k) \right]^L \right\}, \quad (7)$$

where  $\mathbf{K} \equiv \mathbf{k}_1 + \cdots + \mathbf{k}_n$ . Here  $G_+(\bar{\tau}, k)$  represents a bulk-to-boundary propagator connecting the vertex at a time  $\bar{\tau}$  with the boundary at a time  $\tau$ . On the other hand,  $G_{++}(\bar{\tau}, \bar{\tau}, k)$  represents a bulk-to-bulk propagator, in this case connecting the vertex at time  $\bar{\tau}$  with itself. Concretely, it is given by

$$G_{++}(\bar{\tau}, \bar{\tau}, k) = f_k(\bar{\tau}) f_k^*(\bar{\tau}). \quad (8)$$

In Appendix A, we summarize the Feynman rules leading to the expressions we are dealing with. The diagram of Eq. (7) has a symmetry factor  $2^L L!$  due to both the

symmetry of a single loop ( $2^L$ ) and the interchange of loops ( $L!$ ).

Adding up all the diagrams, we then obtain the following full expression for the  $n$ -point correlation function:

$$\langle \varphi_{\mathbf{k}_1} \cdots \varphi_{\mathbf{k}_n} \rangle(\tau) = -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \sum_L \frac{\lambda_{n+2L}^b}{H^4 L!} \times 2 \text{Im} \left\{ \int_{-\infty}^{\tau} \frac{d\bar{\tau}}{\bar{\tau}^4} G_+(\bar{\tau}, k_1) \cdots G_+(\bar{\tau}, k_n) \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^L \right\}, \quad (9)$$

where we have defined  $\sigma_{\text{tot}}^2$  as

$$\sigma_{\text{tot}}^2 \equiv \int \frac{d^3k}{(2\pi)^3} G_{++}(\bar{\tau}, \bar{\tau}, k), \quad (10)$$

which measures the integral contribution of a single loop inside the  $\bar{\tau}$ -integral. At first sight this quantity appears to be time-dependent, implying that the overall dependence of the integrand in Eq. (7) gets more complicated with more and more loops. However, after performing the change of variables  $k \rightarrow p = k/a(\tau)$  from comoving momentum  $k$  to physical momentum  $p$ , one trivially finds that  $\sigma_{\text{tot}}^2$  is a constant. For instance, in the particular case of a massless field in de Sitter spacetime, one finds

$$\sigma_{\text{tot}}^2 = \frac{H^2}{4\pi^2} \int_0^\infty \frac{dp}{p} \left(1 + \frac{p^2}{H^2}\right). \quad (11)$$

Alternatively, for a massive field of mass  $m$ , one finds

$$\sigma_{\text{tot}}^2 = \frac{1}{16\pi H} \int_0^\infty \frac{dp}{p} p^3 \left| H_\nu^{(1)}\left(\frac{p}{H}\right) \right|^2. \quad (12)$$

Consequently, the quantity  $\sigma_{\text{tot}}^{2L}$  can be pulled out of the  $\bar{\tau}$ -integral of Eq. (9) to become an overall constant factor. Of course, despite being a constant,  $\sigma_{\text{tot}}^{2L}$  is clearly divergent due to both the upper and the lower bound of the integral (e.g., in Eq. (11)). In the following two sections, we do not dwell on the specific form of  $\sigma_{\text{tot}}^2$ , and instead examine how to deal with the divergence it implies.

At this point, some readers may object that the time independence of  $\sigma_{\text{tot}}^2$  has been enforced by the change of variables  $k \rightarrow p = k/a(\tau)$ . In Sec. V, we will compute this quantity using dimensional regularization. There, we shall confirm that  $\sigma_{\text{tot}}^2$  is indeed constant, with time-independent divergent pieces that can be absorbed by counterterms in the way we proceed to explain in the following section.

### III. BASIC STRATEGY

Let us split the bare coupling  $\lambda_n^{\text{b}}$  into a renormalized coupling  $\lambda_n$  and a counter-term  $\lambda_n^{\text{ct}}$  as

$$\lambda_n^{\text{b}} = \lambda_n + \lambda_n^{\text{ct}}. \quad (13)$$

More generally, we should also consider normalizing the field  $\varphi$  and split the field normalization analogously. However, given that the diagrams we are examining have loops that carry no external momenta, this is not necessary (i.e., we do not need to worry about renormalization of the kinetic term of the bare theory). Next, given that we are dealing with diagrams with an arbitrary number

of loops, we may further split  $\lambda_n^{\text{ct}}$  into terms of order  $L$ , needed to cancel divergences coming from  $L$ -loop diagrams. Thus we write:

$$\lambda_n^{\text{ct}} = \lambda_n^{(1)} + \lambda_n^{(2)} + \lambda_n^{(3)} + \dots \quad (14)$$

Notationally, we may thus simply write

$$\lambda_n^{\text{b}} = \sum_{L=0}^{\infty} \lambda_n^{(L)}, \quad (15)$$

where  $\lambda_n^{(0)} = \lambda_n$  is just the renormalized coupling already introduced in Eq. (13).

Having defined this splitting, we must re-work the computation of an  $n$ -point correlation function by reorganizing diagrams in terms of the couplings  $\lambda_n^{(L)}$ . To be precise, a diagram with  $L_1$  loops proportional to a counterterm with coupling  $\lambda_{n+2L_1}^{(L_2)}$  must be regarded as of order  $L = L_1 + L_2$ . Then it is a simple matter to see that a contribution to Eq. (6) of order  $L$  is simply obtained by making the following replacement:

$$\lambda_n^{\text{b}} \rightarrow \sum_{s=0}^L \frac{1}{s!} \left(\frac{\sigma_{\text{tot}}^2}{2}\right)^s \lambda_{n+2s}^{(L-s)}. \quad (16)$$

In this way, one finally obtains that the full correlation function, up to first order in  $\lambda$ , has the form

$$\langle \varphi_{\mathbf{k}_1} \dots \varphi_{\mathbf{k}_n} \rangle(\tau) = -(2\pi)^3 \delta(\mathbf{K}) \frac{\lambda_n^{\text{obs}}}{H^4} \times 2 \text{Im} \left\{ \int_{-\infty}^{\tau} \frac{d\bar{\tau}}{\bar{\tau}^4} G_+(\bar{\tau}, k_1) \dots G_+(\bar{\tau}, k_n) \right\}, \quad (17)$$

where

$$\lambda_n^{\text{obs}} \equiv \sum_{L=0}^{\infty} \sum_{s=0}^L \frac{1}{s!} \left(\frac{\sigma_{\text{tot}}^2}{2}\right)^s \lambda_{n+2s}^{(L-s)}. \quad (18)$$

We emphasize that Eq. (17) contains nothing but the same starting expression shown in Eq. (9), obtained after re-organizing counterterms.

The full expression must be finite, and so now the challenge is to determine the appropriate values for the counterterm couplings  $\lambda_n^{(L)}$  (with  $L > 0$ ) in terms of the renormalized couplings  $\lambda_n^{(0)} = \lambda_n$ . As a first approach, let us require that the counterterms  $\lambda_n^{(L)}$  cancel exactly the divergent constant  $\sigma_{\text{tot}}^2$ . This implies the following infinite set of algebraic equations:

$$\sum_{s=0}^L \frac{1}{s!} \left(\frac{\sigma_{\text{tot}}^2}{2}\right)^s \lambda_{n+2s}^{(L-s)} = 0, \quad \text{for all } L > 0. \quad (19)$$

Diagrammatically, the previous set of equations may be summarized as the following cancellation among diagrams with different numbers of loops:

$$\lambda_{n+2L}^{(0)}, \bar{\tau}, L \text{ loops} + \lambda_{n+2(L-1)}^{(1)}, \bar{\tau}, (L-1) \text{ loops} + \dots + \lambda_n^{(L)}, \bar{\tau}, L \text{ loops} = 0. \quad (20)$$

The solution to this set of equations turns out to be (see Appendix B):

$$\lambda_n^{(L)} = (-1)^L \frac{1}{L!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^L \lambda_{n+2L}^{(0)}, \quad (21)$$

which may be proved by induction. With this choice of counterterms, the full expression for the  $n$ -point correlation function in Eq. (17) becomes finite, with  $\lambda_n^{\text{obs}} = \lambda_n^{(0)} = \lambda_n$ . This result, together with the solution in Eq. (21) allows us to express the original divergent bare couplings  $\lambda_n^{\text{b}}$  in terms of the observable couplings  $\lambda_n^{\text{obs}}$  as:

$$\lambda_n^{\text{b}} = \sum_{L=0}^{\infty} (-1)^L \frac{1}{L!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^L \lambda_{n+2L}^{\text{obs}}. \quad (22)$$

It is then natural to define the observable potential  $\mathcal{V}_{\text{obs}}(\varphi)$  as that obtained by resumming the Taylor coefficients  $\lambda_n^{\text{obs}} = \lambda_n^{(0)}$ :

$$\mathcal{V}_{\text{obs}}(\varphi) \equiv \sum_{n=0}^{\infty} \frac{\lambda_n^{\text{obs}}}{n!} \varphi^n. \quad (23)$$

Then, thanks to Eq. (22), it is possible to verify that  $\mathcal{V}_{\text{obs}}(\varphi)$  and  $\mathcal{V}_{\text{b}}(\varphi)$  are related with the help of a Weierstrass transformation of the form

$$\mathcal{V}_{\text{b}}(\varphi) = e^{-\frac{1}{2}\sigma_{\text{tot}}^2 \frac{\partial^2}{\partial \varphi^2}} \mathcal{V}_{\text{obs}}(\varphi). \quad (24)$$

In this way, observables, up to order  $\lambda$ , will depend on the observable potential  $\mathcal{V}_{\text{obs}}(\varphi)$ , and not the formally infinite potential  $\mathcal{V}_{\text{b}}(\varphi)$ . This agrees with the results of Refs. [27, 29].

#### IV. RENORMALIZATION SCHEME

There must exist some arbitrariness in the choice of the counterterm couplings  $\lambda_n^{(L)}$ , for  $L > 0$ , as usual in standard perturbation theory. For instance, in dimensional regularization,  $\sigma_{\text{tot}}^2$  might have a divergent piece proportional to  $\delta^{-1}$  (where  $\delta$  parametrizes the departure from 3 spatial dimensions), together with a finite part. Thus, if we aim for a minimal subtraction scheme, we should tune the counterterms  $\lambda_n^{(L)}$  to be proportional to  $\delta^{-L}$  in order to capture only the divergent contributions.

To proceed, let us split  $\sigma_{\text{tot}}^2$  into two terms as  $\sigma_{\text{tot}}^2 = \sigma_0^2 + \sigma_{\infty}^2$ , and choose the counterterms  $\lambda_n^{(L)}$  in such a way

that they cancel out the divergent piece, say,  $\sigma_{\infty}^2$ . In this case, the effective coupling can be written as

$$\lambda_n^{\text{obs}} = \sum_{L=0}^{\infty} \sum_{s=0}^L \frac{1}{s!} \left( \frac{1}{2}\sigma_0^2 + \frac{1}{2}\sigma_{\infty}^2 \right)^s \lambda_{n+2s}^{(L-s)}. \quad (25)$$

Now, motivated by the solution in Eq. (21), let us guess that the correct choice for the counterterm couplings  $\lambda_n^{(L)}$  (for  $L > 0$ ) is given by

$$\lambda_n^{(L)} = (-1)^L \frac{1}{L!} \left( \frac{1}{2}\sigma_{\infty}^2 \right)^2 \lambda_{n+2L}^{(0)}. \quad (26)$$

By plugging this ansatz back into Eq. (25), we are led to

$$\lambda_n^{\text{obs}} = \sum_{L=0}^{\infty} \frac{\lambda_{n+2L}^{(0)}}{L!} \sum_{s=0}^L \frac{L!}{s!(L-s)!} \times \left( -\frac{1}{2}\sigma_{\infty}^2 \right)^{L-s} \left( \frac{1}{2}\sigma_0^2 + \frac{1}{2}\sigma_{\infty}^2 \right)^s. \quad (27)$$

Then, by recognising the general formula  $(x+y)^L = \sum_{s=0}^L \frac{L!}{s!(L-s)!} x^{L-s} y^s$ , we see that:

$$\lambda_n^{\text{obs}} = \sum_{L=0}^{\infty} \frac{\lambda_{n+2L}^{(0)}}{L!} \left( \frac{1}{2}\sigma_0^2 \right)^L. \quad (28)$$

Thus, with the choice in Eq. (26), one can remove the specific divergent part of  $\sigma_{\text{tot}}^2$  and retain the finite contribution.

Let us now define the renormalized potential  $\mathcal{V}_{\text{ren}}(\varphi)$  as the resulting resummation of the Taylor coefficients  $\lambda_n^{(0)}$ :

$$\mathcal{V}_{\text{ren}}(\varphi) \equiv \sum_{n=0}^{\infty} \frac{\lambda_n^{(0)}}{n!} \varphi^n. \quad (29)$$

Notice that in Sec. III,  $\mathcal{V}_{\text{ren}}(\varphi)$  and  $\mathcal{V}_{\text{obs}}(\varphi)$  coincided because  $\lambda_n^{\text{obs}} = \lambda_n^{(0)}$ . Here, given that  $\sigma_0^2 \neq 0$ , Eq. (28) implies that

$$\mathcal{V}_{\text{obs}}(\varphi) = e^{\frac{\sigma_0^2}{2} \frac{\partial^2}{\partial \varphi^2}} \mathcal{V}_{\text{ren}}(\varphi). \quad (30)$$

On the other hand, from Eq. (26), we see that

$$\mathcal{V}_{\text{b}}(\varphi) = e^{-\frac{\sigma_{\infty}^2}{2} \frac{\partial^2}{\partial \varphi^2}} \mathcal{V}_{\text{ren}}(\varphi). \quad (31)$$



Both Eqs. (30) and (31) directly imply  $\mathcal{V}_{\text{obs}}(\varphi) = e^{\frac{\sigma_{\text{tot}}^2}{2} \frac{\partial^2}{\partial \varphi^2}} \mathcal{V}_{\text{b}}(\varphi)$ , which is the result previously derived in Sec. III. Thus, the arbitrariness in the choice of the counterterms  $\lambda_n^{(L)}$  is reflected in the arbitrary parameter  $\sigma_0^2$  connecting the observable potential  $\mathcal{V}_{\text{obs}}(\varphi)$  and the renormalized potential  $\mathcal{V}_{\text{ren}}(\varphi)$ . As usual, the observable potential  $\mathcal{V}_{\text{obs}}(\varphi)$  cannot depend on this arbitrary parameter, and, as such, the renormalized potential  $\mathcal{V}_{\text{ren}}(\varphi)$  must depend on  $\sigma_0^2$  in such a way that it cancels out after the Weierstrass operator  $\exp\{\frac{1}{2}\sigma_0^2\partial_\varphi^2\}$  acts on it. In the following section, we discuss the renormalization flow that this parameter  $\sigma_0^2$  implies.

## V. DIMENSIONAL REGULARIZATION

To put our approach in more concrete terms, let us analyze the divergent integrals within dimensional regularization. To proceed, we must first extend some relevant quantities from  $d = 3$  to an arbitrary spatial dimension  $d = 3 + \delta$ . To start with, notice that in 3 dimensions the couplings  $\lambda_n$  have mass-dimension  $[\lambda_n] = 4 - n$ . We may conveniently keep  $[\lambda_n]$  fixed in arbitrary spatial dimension  $d = 3 + \delta$  by performing the following replacement:

$$\lambda_n \rightarrow \lambda_n \tilde{\mu}^{\delta(1-n/2)}, \quad (32)$$

where  $\tilde{\mu}$  is the usual renormalization mass scale. The field has mass-dimension  $[\varphi] = 1 + \delta/2$ , which in turn implies that  $f_k(\tau)$  has a fixed mass-dimension of  $-1/2$ . For this reason, it is convenient to redefine  $\sigma_{\text{tot}}^2$  in a way that its

mass-dimension remains fixed to 2. The appropriate redefinition anticipates the relation between the couplings  $\lambda_n$  and  $\sigma_{\text{tot}}^2$  presented in Eq. (18):

$$\sigma_{\text{tot}}^2 \equiv \frac{1}{\tilde{\mu}^\delta} \int \frac{d^{3+\delta}k}{(2\pi)^{3+\delta}} f_k(\tau) f_k^*(\tau). \quad (33)$$

The equation of motion respected by  $f_k(\tau)$  in  $d = 3 + \delta$  dimensions is given by:

$$f_k'' - \frac{2 + \delta}{\tau} f_k' + \left( k^2 + \frac{m^2}{H^2 \tau^2} \right) f_k = 0. \quad (34)$$

The solution to this equation with appropriate Bunch-Davies initial conditions leads to [37]

$$f_k(\tau) = \frac{i\sqrt{\pi}H^{1+\delta/2}}{2} (-\tau)^{(3+\delta)/2} H_\nu^{(1)}(-k\tau), \quad (35)$$

where  $H_\nu^{(1)}(x)$  is a Hankel function of the first kind with parameter  $\nu = \sqrt{(3 + \delta)^2/4 - m^2/H^2}$ . We may now compute  $\sigma_{\text{tot}}^2$  by substituting Eq. (35) back into Eq. (33); one finds:

$$\sigma_{\text{tot}}^2 = \frac{\pi^{-(1+\delta)/2} H^{2+\delta}}{16\Gamma(\frac{3+\delta}{2})(2\tilde{\mu})^\delta} \int_0^\infty \frac{dx}{x} x^{3+\delta} |H_\nu^{(1)}(x)|^2, \quad (36)$$

where we made the change of variables  $x = -k\tau$ . Notice that, just as discussed in Sec. II, the loop factor  $\sigma_{\text{tot}}^2$  is now explicitly time independent. What remains now is to integrate this expression, which is possible as long as  $\delta < -2$  and  $2\nu + \delta > -3$ . Where these conditions are satisfied, the integral results in:

$$\sigma_{\text{tot}}^2 = \frac{\pi^{-\delta/2}}{16\Gamma(\frac{3+\delta}{2})(2\tilde{\mu})^\delta} \frac{H^{2+\delta} \cos[\pi\nu] \csc[\pi(3+\delta)/2] \csc[\pi((3+\delta)/2 - \nu)] \csc[\pi((3+\delta)/2 + \nu)] \Gamma[-(2+\delta)/2]}{\Gamma[-(1+\delta)/2] \Gamma[-(1+\delta)/2 - \nu] \Gamma[-(1+\delta)/2 + \nu]}. \quad (37)$$

We may now analytically continue this result for all values of  $\delta$ . The quantity  $\sigma_{\text{tot}}^2$  is a complicated function of  $\delta$  and  $\nu$  in the vicinity of  $(\delta, \nu) = (0, 3/2)$ , and the divergent nature of  $\sigma_{\text{tot}}^2$  depends on how we choose to approach this point. One way to deal with this complication, as explored in Refs. [48, 49], consists of fixing the mass parameter  $m^2$  appearing in Eq. (34) as  $m^2 = m_{\text{ph}}^2 + H^2(d^2 - 9)/4$ , with  $m_{\text{ph}}$  identified as the physical mass of  $\varphi$  in the free-theory case, where  $\mathcal{V}(\varphi) = 0$ . This choice ensures that  $\nu = \sqrt{9/4 - m_{\text{ph}}^2/H^2}$  for all values of  $d$ , and the theory remains scale invariant in the vicinity of  $d = 3$  for the particular case of a massless field ( $m_{\text{ph}} = 0$ ). With this choice, it is then possible to verify that

$$\sigma_{\text{tot}}^2 \rightarrow 0 \quad \text{for} \quad \nu \rightarrow 3/2, \quad (38)$$

which corresponds to the limit  $m_{\text{ph}} \rightarrow 0$ , independent of the value of  $\delta$  [49, 50]. This is simply because the  $x$ -integral of Eq. (36) becomes  $\int_0^\infty dx x^\delta (1+x^2)$ , which, be-

ing the integral of a polynomial, vanishes in dimensional regularization [51].

Instead of fixing the mass parameter  $m^2$  in the way just described, we could simply expand  $\sigma_{\text{tot}}^2$  first in terms of  $m^2/H^2$  and then in terms of  $\delta$ . One then finds:

$$\sigma_{\text{tot}}^2 = -\frac{H^2}{4\pi^2\delta} + \frac{3H^4}{8\pi^2m^2} + \frac{H^2}{4\pi^2} \ln\left(\frac{\mu}{H}\right) + \mathcal{O}\left(\delta, \frac{m^2}{H^2}\right), \quad (39)$$

where we have redefined  $\mu \equiv \sqrt{4\pi}e^{(\gamma_E/2-7/4)}\tilde{\mu}$ , with  $\gamma_E$  the Euler-Mascheroni constant. The previous result explicitly displays two divergent contributions, one from  $\delta \rightarrow 0$ , which may be identified with an ultraviolet divergence, and a second one, from  $m^2/H^2 \rightarrow 0$ , which embodies the infrared divergent part. Of course, these divergences are not a problem as we can now proceed to minimally subtract them in the way explained in the previous section. We end up finding a finite contribution

$\sigma_0^2$  for loop integrals, given by:

$$\sigma_0^2 = \frac{H^2}{4\pi^2} \ln\left(\frac{\mu}{H}\right). \quad (40)$$

The arbitrariness of the subtraction, as usual, is reflected in the dependence of the remaining finite part on the renormalization scale  $\mu$ . The fact that the loops discussed here have no physical effect aligns with the considerations of Refs. [49, 50], which also perform dimensional regularization.

As commented in Sec. IV, the observable potential (i.e., the potential defining the couplings appearing in observable correlation functions) must be independent of  $\mu$ . Thus, from Eq. (40), together with Eq. (30), we see that the renormalized potential must satisfy the following renormalization group flow equation:

$$\mu \frac{\partial}{\partial \mu} \mathcal{V}_{\text{ren}}(\varphi) = -\frac{H^2}{8\pi^2} \mathcal{V}_{\text{ren}}''(\varphi). \quad (41)$$

One may put this result in concrete terms by considering an axionic potential  $\mathcal{V}_{\text{ren}}(\varphi) = -A \cos(\varphi/f)$ , with  $f$  the axion decay constant. Then, from Eq. (30), it follows that the observable potential is read as  $\mathcal{V}_{\text{obs}}(\varphi) = -A_{\text{obs}} \cos(\varphi/f)$  with  $A_{\text{obs}}$  and  $A$  related as

$$A_{\text{obs}} = A(\mu) e^{-\frac{1}{2} \frac{\sigma_0^2(\mu)}{f^2}}. \quad (42)$$

Because  $A_{\text{obs}}$  cannot depend on the renormalization scale, we see that  $A(\mu) \propto \exp\left[\frac{H^2}{8\pi^2 f^2} \ln\left(\frac{\mu}{H}\right)\right]$ . On the other hand, the decay coupling constant  $f$  does not receive loop corrections.

Given that observables to order  $\lambda$  do not have external momenta running through loops, we do not see combinations such as  $\ln(k/\mu)$ , and so the renormalization scale  $\mu$  is not a useful quantity allowing one to assess the flow of momenta through couplings. Nevertheless, in the next section, we will encounter similar expressions involving cutoff momenta that will allow us to assess the validity of perturbation theory of order  $\lambda$  at different scales.

## VI. WILSONIAN APPROACH

Let us consider the problem of computing correlation functions within an effective field theory point of view. Our starting point is an effective field theory defined over a range of physical momenta bounded by infrared and ultraviolet cutoff scales  $\Lambda_{\text{IR}}$  and  $\Lambda_{\text{UV}}$ , respectively:

$$S_{\text{eff}}(\Lambda_{\text{IR}}, \Lambda_{\text{UV}}) = \int d^3x d\tau a^4 \left[ \frac{(\varphi')^2}{2a^2} - \frac{(\nabla\varphi)^2}{2a^2} - \mathcal{V}_{\text{eff}}(\varphi) \right]. \quad (43)$$

By definition, the Wilsonian coefficients in  $S_{\text{eff}}$  incorporate the integration of momenta below  $\Lambda_{\text{IR}}$  and above  $\Lambda_{\text{UV}}$ , to all order with respect to both vertices and loops.

As a consequence, the computation of correlation functions, including loop corrections, must be finite. By writing  $\mathcal{V}_{\text{eff}}(\varphi) = \sum_n \frac{\lambda_n^{\text{eff}}}{n!} \varphi^n$ , a direct computation to first order with respect to  $\lambda_n^{\text{eff}}$  leads to

$$\langle \varphi_{\mathbf{k}_1} \cdots \varphi_{\mathbf{k}_n} \rangle(\tau) = -\lambda_n^{\text{obs}} 2 \text{Im} \left\{ \int_{-\infty}^{\tau} \frac{d\bar{\tau}}{\bar{\tau}^4} G_+(\bar{\tau}, k_1) \cdots G_+(\bar{\tau}, k_n) \right\}, \quad (44)$$

where  $\lambda_n^{\text{obs}}$  is found to be:

$$\lambda_n^{\text{obs}} = \sum_L \frac{1}{L!} \lambda_{n+2L}^{\text{eff}} \left[ \frac{1}{2} \sigma^2(\Lambda_{\text{IR}}, \Lambda_{\text{UV}}) \right]^L. \quad (45)$$

Of course,  $\lambda_n^{\text{obs}}$  coincides with the same observable couplings encountered in the previous sections. Here,  $\sigma^2(\Lambda_{\text{IR}}, \Lambda_{\text{UV}})$  is nothing but the loop integral in Eq. (10) evaluated within the momentum range bounded by  $\Lambda_{\text{IR}}$  and  $\Lambda_{\text{UV}}$ . In the particular case of a massless field, one explicitly finds:

$$\begin{aligned} \sigma^2(\Lambda_{\text{IR}}, \Lambda_{\text{UV}}) &= \frac{H^2}{4\pi^2} \int_{\Lambda_{\text{IR}}}^{\Lambda_{\text{UV}}} \frac{dp}{p} \left( 1 + \frac{p^2}{H^2} \right) \\ &= \frac{H^2}{4\pi^2} \left[ \ln \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}} + \frac{\Lambda_{\text{UV}}^2 - \Lambda_{\text{IR}}^2}{2H^2} \right]. \end{aligned} \quad (46)$$

Now, from Eq. (45), it is direct to see that

$$\mathcal{V}_{\text{obs}}(\varphi) = e^{\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial \varphi^2}} \mathcal{V}_{\text{eff}}(\varphi), \quad (47)$$

where  $\sigma^2$  stands for  $\sigma^2(\Lambda_{\text{IR}}, \Lambda_{\text{UV}})$ .

The effective potential  $\mathcal{V}_{\text{eff}}(\varphi)$  must depend on the cutoff scales  $\Lambda_{\text{IR}}$  and  $\Lambda_{\text{UV}}$  in order to compensate for the independence of the observable potential  $\mathcal{V}_{\text{obs}}(\varphi)$  on such arbitrary quantities. Notably, we see that the logarithmic dependence on the ratio  $\Lambda_{\text{UV}}/\Lambda_{\text{IR}}$  coincides with that of the renormalization scale  $\mu$  found in the previous section. Given that the renormalization scale  $\mu$  is arbitrary, the result of this section can be reconciled with those found in the previous sections through the identification  $\frac{\mu}{H} \rightarrow \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}} e^{\frac{1}{2H^2}(\Lambda_{\text{UV}}^2 - \Lambda_{\text{IR}}^2)}$ . Thus, Eq. (47) determines a flow equation for  $\mathcal{V}_{\text{eff}}(\varphi)$  in terms of both  $\Lambda_{\text{IR}}$  and  $\Lambda_{\text{UV}}$ , in the same manner as in Eq. (41).

In order to be able to perform perturbative computations with this effective theory,  $\mathcal{V}_{\text{eff}}(\varphi)$  must respect the following perturbative condition:

$$|\mathcal{V}_{\text{eff}}(\varphi)| \ll H^4. \quad (48)$$

Hence, the flow of  $\mathcal{V}_{\text{eff}}(\varphi)$  in terms of the cutoff scales could break this perturbative condition for certain values of  $\Lambda_{\text{IR}}$  and  $\Lambda_{\text{UV}}$ . To appreciate the consequence of this condition, let us again resort to the example of an axionic potential:

$$\mathcal{V}_{\text{eff}}(\varphi) = -A \cos(\varphi/f). \quad (49)$$

The perturbative condition requires  $A \ll H^4$ . At the same time, thanks to Eq. (47), the observable potential can be expressed in terms of  $\mathcal{V}_{\text{eff}}(\varphi)$  as:

$$\mathcal{V}_{\text{obs}}(\varphi) = -Ae^{-\frac{\sigma^2}{2f^2}} \cos(\varphi/f). \quad (50)$$

This implies that the amplitude of the effective potential must depend on the cutoff scales as  $A = A_{\text{obs}}e^{\frac{\sigma^2}{2f^2}}$ , where  $A_{\text{obs}}$  is the observable amplitude. Now, the perturbative condition on the effective potential reads  $\frac{A_{\text{obs}}}{H^4}e^{\frac{\sigma^2}{2f^2}} \ll 1$ , which implies that the theory becomes strongly coupled for values

$$\sigma^2(\Lambda_{\text{IR}}, \Lambda_{\text{UV}}) \sim 2f^2 \ln\left(\frac{H^4}{A_{\text{obs}}}\right). \quad (51)$$

This expression allows us to assess at what infrared or ultraviolet scales the theory becomes non-perturbative. For instance, if we fix  $\Lambda_{\text{UV}}$  to values not far from  $H$ , and push  $\Lambda_{\text{IR}}$  well below  $H$ , then  $\sigma^2 \sim H^2 \ln\left(\frac{H}{\Lambda_{\text{IR}}}\right)$  and one learns that the theory becomes non-perturbative at the infrared scale

$$\Lambda_{\text{IR}} \sim He^{-8\pi\frac{f^2}{H^2} \ln\left(\frac{H^4}{A_{\text{obs}}}\right)}. \quad (52)$$

For momenta below this value of  $\Lambda_{\text{IR}}$ , one cannot trust perturbative computations performed with the effective theory at hand.

## VII. DISCUSSION

To verify the infrared behavior of the correlation function  $\langle \varphi_{\mathbf{k}_1} \cdots \varphi_{\mathbf{k}_n} \rangle(\tau)$  of Eq. (17), we may evaluate it in the limit of superhorizon external momenta  $k_i|\tau| \ll 1$ . To this end, it is convenient to split the time integral by introducing a fiducial time  $\tau_0 < \tau$ . In this way, one verifies that  $k_i|\tau_0| \ll 1$ , and, given that the integrand for times  $-\infty < \bar{\tau} < \tau_0$  is highly oscillatory, its integration yields a convergent piece. On the contrary, the integral over the range  $\tau_0 < \bar{\tau} < \tau$  becomes dominated by a logarithmic contribution with respect to  $\tau$ . One finds:

$$\begin{aligned} \langle \varphi_{\mathbf{k}_1} \cdots \varphi_{\mathbf{k}_n} \rangle(\tau) &= -(2\pi)^3 \delta^{(3)}(\mathbf{K}) \\ &\times \lambda_n^{\text{obs}} \frac{H^{2(n-2)}}{3 \times 2^{n-1}} \frac{k_1^3 + \cdots + k_n^3}{k_1^3 \cdots k_n^3} \ln\left(\frac{\tau_0}{\tau}\right) + \cdots, \end{aligned} \quad (53)$$

where the ellipses denote the convergent terms coming from the integration over the range  $-\infty < \bar{\tau} < \tau_0$ . Of course, the full correlation function is independent of the arbitrary time  $\tau_0$ , and one may simply replace  $\ln(\tau_0/\tau)$  by  $\ln a(\tau)$ . This result confirms that correlation functions in comoving momentum space, up to first order in the number of vertices, have an asymptotic divergence as  $\tau \rightarrow 0$  proportional to  $\ln a(\tau)$ .

To appreciate the importance of the present claims, let us examine the computation of  $n$ -cumulants<sup>4</sup> of  $\varphi$  on superhorizon scales. As already mentioned in the introduction, observers are limited to a finite range of scales. Consequentially, correlation functions in coordinate space are obtained from correlators in momentum space with the help of window functions filtering the observable range of scales. Consider, for instance, the definition of the so-called infrared field [5] as

$$\varphi_L(\mathbf{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \varphi_{\mathbf{k}}(\tau) W(k, \tau), \quad (54)$$

where the window function  $W(k, \tau)$ , designed to select superhorizon scales  $k/a(t) \ll H$ , is defined in the following way:

$$W(k, \tau) \equiv \theta(k_*(\tau) - k) \times \theta(k - k_*(\tau_1)), \quad (55)$$

where  $\theta(x)$  is the Heaviside step function and  $k_*(\tau) = \alpha H a(\tau)$ , for a small dimensionless parameter  $\alpha > 0$ . Note that  $k_*(\tau_1)$  is a comoving infrared cutoff that limits the number of modes involved in the infrared field in such a way that, at  $\tau = \tau_i$ , the range of momenta defining  $\varphi_L(\mathbf{x}, \tau)$  is empty. For  $\tau > \tau_i$ , the range starts to grow as more and more modes enter the definition of  $\varphi_L(\mathbf{x}, \tau)$ , implying that the variance  $\sigma_L^2(\tau) \equiv \langle \varphi_L(\mathbf{x}, \tau) \varphi_L(\mathbf{y}, \tau) \rangle_{\mathbf{x}=\mathbf{y}}$ , at zeroth order, grows proportionally to  $\ln(a(\tau)/a(\tau_1))$ :

$$\sigma_L^2(\tau) = \frac{H^2}{4\pi^2} \ln\left(\frac{a(\tau)}{a(\tau_1)}\right). \quad (56)$$

Using Eq. (53), we can now compute  $n$ -cumulants to first order in the potential  $\mathcal{V}(\varphi)$ . One finds

$$\langle \varphi_L^n(\tau) \rangle_c = -\frac{4\pi^2 n \lambda_n^{\text{obs}}}{3H^4} \sigma_L^{2n}(\tau). \quad (57)$$

In contrast, if one uses comoving cutoffs to compute correlation functions, loop integrals would modify the time-dependence of these correlators in a way that is sensitive to the number of loops. In fact, one finds:

$$\langle \varphi_L^n(\tau) \rangle_c = -\frac{4\pi^2 n}{3H^4} \sigma_L^{2n}(\tau) \sum_L \frac{\lambda_{n+2L}}{(n+L)L!} \left(\frac{\sigma_L^2(\tau)}{2}\right)^L, \quad (58)$$

where  $\lambda_n$  represent ultraviolet (but not infrared) renormalized couplings, and the index  $L$ , over which the sum is performed, informs us about the number of loops contributing to a given term. All in all, because  $\langle \varphi^n(\tau) \rangle_c$  is an observable, it is pertinent to establish the correct procedure to determine the time dependence of  $n$ -point correlation functions implied by loops. In Ref. [30], the consequence emerging from the difference between Eqs. (57) and (58) is examined within the context of the stochastic approach to inflation [52, 53].

<sup>4</sup> Here, an  $n$ -cumulant  $\langle \varphi^n(\tau) \rangle_c$  is the connected part of an  $n$ -point correlation function in coordinate space evaluated at the coincident limit, where all the coordinates are equal.



## VIII. CONCLUSIONS

Within the dimensional regularization framework, correlation functions to first order in the scalar field potential  $\mathcal{V}(\varphi)$ , and to all orders in loops, are indistinguishable from their tree-level counterparts. To reach this conclusion, we did not consider cutoffs in momentum space, either ultraviolet or infrared. In contrast, if one enforces the introduction of an infrared comoving cutoff, inevitably, the correlation function will pick up a time dependence sensitive to the number of loops participating in its computation. In other words, working with comoving cutoffs introduces a strong dependence of the correlation function on the cutoff of the theory. But, let us emphasize, this dependence is not seen if one directly integrates the loop within the dimensional regularization scheme.

The single-vertex result presented here is suggestive of the more general case with diagrams with two or more vertices: Loops do not modify the leading dependence of diagrams on powers of  $\ln a(\tau)$ . Of course, the analysis of diagrams with two or more vertices is much more complicated as loops carry external momenta. However, from the present analysis it should be clear that their treatment should not require the introduction of comoving cutoffs. More precisely, loop integrals should be performed within the dimensional regularization scheme, with ultraviolet and infrared divergences treated in the standard way. A promising avenue to settle the situation with more complicated diagrams is offered by recent developments of techniques to integrate general diagrams in de Sitter spacetime [54–56].

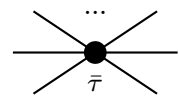
Naturally, the class of theories we are dealing with, being non-renormalizable, will require the introduction of terms (possibly non-local) in order to eliminate infrared divergences without breaking the crucial property of time translation invariance found in the starting theory. This breaking should only be reflected in correlation functions [1–3] due to the use of the Bunch-Davies state but not in the Lagrangian. This is in contrast with earlier works featuring dimensional regularization, which result in counterterms explicitly breaking de Sitter invariance. Note that here we are in the rigid de Sitter limit, controlled by the perturbative condition in Eq. (48), which differs from the case of inflation where time-translation invariance is lost [57] as a result of a preferred time set by the slow-rolling inflaton.

## ACKNOWLEDGMENTS

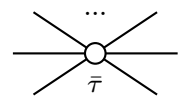
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## Appendix A: Feynman rules for correlation functions

In this appendix, we summarize the Feynman rules allowing one to obtain a correlation function  $\langle \varphi^n(\mathbf{k}_1, \dots, \mathbf{k}_n) \rangle$  evaluated at a given time  $\tau$ . We will restrict our attention to a theory of the form in Eq. (1). To start with, a single term of the expansion in Eq. (5), proportional to  $\lambda_n$ , defines two classes of vertices, hereby distinguished by black and white solid dots:



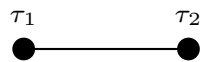
$$-i\lambda_n \int_{-\infty}^{\tau} d\bar{\tau} a^4(\bar{\tau}) (2\pi)^3 \delta^{(3)}\left(\sum_i k_i\right) \left[ \dots \right], \quad (\text{A1})$$



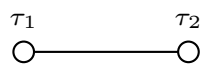
$$+i\lambda_n \int_{-\infty}^{\tau} d\bar{\tau} a^4(\bar{\tau}) (2\pi)^3 \delta^{(3)}\left(\sum_i k_i\right) \left[ \dots \right], \quad (\text{A2})$$

where  $\delta^{(3)}(\sum_i k_i)$  denotes a Dirac delta function enforcing conservation of momenta flowing into the vertex through the legs. Each vertex is characterized by a time label  $\bar{\tau}$  that must be integrated from  $-\infty$  up until the time  $\tau$  at which  $n$ -point correlation functions are evaluated. The square brackets on the right hand side of the previous equations indicate that any function of  $\bar{\tau}$  must be integrated in this way.

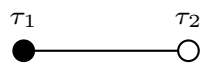
Given that we have two classes of vertices, the theory will contain four types of internal propagators. The corresponding internal propagators are given by



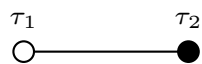
$$G_{++}(k, \tau_1, \tau_2), \quad (\text{A3})$$



$$G_{--}(k, \tau_1, \tau_2), \quad (\text{A4})$$



$$G_{+-}(k, \tau_1, \tau_2), \quad (\text{A5})$$



$$G_{-+}(k, \tau_1, \tau_2). \quad (\text{A6})$$

The analytical expressions for the quantities appearing at the right hand side of the previous assignments are given by:

$$G_{++}(k, \tau_1, \tau_2) = f(k, \tau_1) f^*(k, \tau_2) \theta(\tau_1 - \tau_2) + f^*(k, \tau_1) f(k, \tau_2) \theta(\tau_2 - \tau_1), \quad (\text{A7})$$

$$G_{--}(k, \tau_1, \tau_2) = f^*(k, \tau_1) f(k, \tau_2) \theta(\tau_1 - \tau_2) + f(k, \tau_1) f^*(k, \tau_2) \theta(\tau_2 - \tau_1), \quad (\text{A8})$$

$$G_{+-}(k, \tau_1, \tau_2) = f^*(k, \tau_1) f(k, \tau_2), \quad (\text{A9})$$

$$G_{-+}(k, \tau_1, \tau_2) = f(k, \tau_1) f^*(k, \tau_2). \quad (\text{A10})$$

Just as in the case of vertices, it can be verified explicitly that propagators with given colors (black or white)

at their ends are the complex conjugates of propagators with the opposite colors.

Furthermore, the vertices (evaluated at times  $\tau_1, \tau_2, \tau_3$ , etc.) must be connected to a surface labelled with the time  $\tau$  through bulk-to-boundary propagators. These receive the following assignments:

$$\begin{array}{c} \bar{\tau} \\ \bullet \end{array} \text{---} \begin{array}{c} \tau \\ \square \end{array} \longrightarrow G_+(k, \bar{\tau}), \quad (\text{A11})$$

$$\begin{array}{c} \bar{\tau} \\ \circ \end{array} \text{---} \begin{array}{c} \tau \\ \square \end{array} \longrightarrow G_-(k, \bar{\tau}). \quad (\text{A12})$$

Note that it is unnecessary to assign colors to the square defining the end point evaluated at  $\tau$ . The analytical

expressions for these two bulk-to-boundary propagators read:

$$G_+(k, \bar{\tau}) = f^*(k, \bar{\tau})f(k, \tau), \quad (\text{A13})$$

$$G_-(k, \bar{\tau}) = f(k, \bar{\tau})f^*(k, \tau). \quad (\text{A14})$$

Every internal  $k$  momentum flowing through internal propagators must be integrated with  $\int \frac{d^3k}{(2\pi)^3}$ . Then, the correlation function  $\langle \varphi^n(\mathbf{k}_1, \dots, \mathbf{k}_n) \rangle$  corresponds to the summation of every diagram with  $n$  external legs, truncated to the desired order.

To finish, let us consider a general 1-vertex diagram with  $n$ -external lines and  $L$  loops. This should be written as the sum of two diagrams with black and white vertices respectively:

$$\begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \tau \text{---} \square \text{---} \dots \text{---} \square \text{---} \tau \\ \lambda_{n+2L}^b, \bar{\tau} \\ \cdot \cdot \cdot L \text{ loops} \end{array} = \begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \tau \text{---} \square \text{---} \dots \text{---} \square \text{---} \tau \\ \lambda_{n+2L}^b, \bar{\tau} \\ \cdot \cdot \cdot L \text{ loops} \end{array} + \begin{array}{c} k_1 \quad k_2 \quad \dots \quad k_{n-1} \quad k_n \\ \tau \text{---} \square \text{---} \dots \text{---} \square \text{---} \tau \\ \lambda_{n+2L}^b, \bar{\tau} \\ \cdot \cdot \cdot L \text{ loops} \end{array} \quad (\text{A15})$$

Following the rules just outlined, and taking into account the appropriate symmetry factors, one then obtains the expression presented in Eq. (7).

Written more explicitly, this system has the form:

## Appendix B: Proof of Equation (21)

Here, we show how to obtain the solution in Eq. (21) to the system of algebraic equations in Eq. (19) via induc-

$$\lambda_n^{(1)} + \left( \frac{\sigma_{\text{tot}}^2}{2} \right) \lambda_{n+2}^{(0)} = 0, \quad (\text{B1})$$

$$\lambda_n^{(2)} + \left( \frac{\sigma_{\text{tot}}^2}{2} \right) \lambda_{n+2}^{(1)} + \frac{1}{2!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^2 \lambda_{n+4}^{(0)} = 0, \quad (\text{B2})$$

$$\lambda_n^{(3)} + \left( \frac{\sigma_{\text{tot}}^2}{2} \right) \lambda_{n+2}^{(2)} + \frac{1}{2!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^2 \lambda_{n+4}^{(1)} + \frac{1}{3!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^3 \lambda_{n+6}^{(0)} = 0, \quad (\text{B3})$$

⋮

$$\lambda_n^{(L)} + \left( \frac{\sigma_{\text{tot}}^2}{2} \right) \lambda_{n+2}^{(L-1)} + \frac{1}{2!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^2 \lambda_{n+4}^{(L-2)} + \dots + \frac{1}{L!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^L \lambda_{n+2L}^{(0)} = 0, \quad (\text{B4})$$

and so on and so forth. The first equation (Eq. (B1)) has the solution  $\lambda_n^{(1)} = -\left( \frac{\sigma_{\text{tot}}^2}{2} \right) \lambda_{n+2}^{(0)}$ . Thus, let us as-

sume that  $\lambda_n^{(p)} = (-1)^p \frac{1}{p!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^p \lambda_{n+2p}^{(0)}$  for all  $p \leq L-1$  and show that it also holds true for  $p = L$ . To proceed, notice that  $\lambda_n^{(L)}$  satisfies Eq. (B4), where every

$\lambda_n^{(p)}$  with  $p \leq L-1$  is known. Then, replacing  $\lambda_{n+2(L-p)}^{(p)} = (-1)^p \frac{1}{p!} \left(\frac{\sigma_{\text{tot}}^2}{2}\right)^p \lambda_{n+2L}^{(0)}$  for  $p \leq L-1$ , Eq. (B4) gives us back:

$$\lambda_n^{(L)} = - \left( \sum_{p=0}^{L-1} (-1)^p \frac{1}{(L-p)!p!} \right) \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^L \lambda_{n+2L}^{(0)}. \quad (\text{B5})$$

Thanks to Pascal's formula for binomial coefficients, it is possible to show that:

$$\sum_{p=0}^{L-1} (-1)^p \frac{1}{(L-p)!p!} = (-1)^{L-1} \frac{1}{L!}. \quad (\text{B6})$$

This finally leads to the result in Eq. (21):

$$\lambda_n^{(L)} = (-1)^L \frac{1}{L!} \left( \frac{\sigma_{\text{tot}}^2}{2} \right)^L \lambda_{n+2L}^{(0)}. \quad (\text{B7})$$

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